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Matrix models, monopoles and modified moduli

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Abstract: Motivated by the Dijkgraaf-Vafa correspondence, we consider the matrix model duals of $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N_c)$ gauge theories with N_f flavors. We demonstrate via the matrix model solutions a relation between vacua of theories with different numbers of colors and flavors. This relation is due to an $\mathcal{N}=2$ nonrenormalization theorem which is inherited by these $\mathcal{N}=1$ theories. Specializing to the case $N_f=N_c$, the simplest theory containing baryons, we demonstrate that the explicit matrix model predictions for the locations on the Coulomb branch at which monopoles condense are consistent with the quantum modified constraints on the moduli in the theory. The matrix model solutions include the case that baryons obtain vacuum expectation values. In specific cases we check explicitly that these results are also consistent with the factorization of corresponding Seiberg-Witten curves. Certain results are easily understood in terms of M5-brane constructions of these gauge theories.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality.

Contents

1. Introduction

There has been a recent resurgence of interest in supersymmetric (SUSY) gauge theories, due in part to the observation of Dijkgraaf and Vafa [1] that certain classes of supersymmetric theories can be described by matrix models. This correspondence exists for pure $\mathcal{N} = 2$ SUSY U(N) gauge theory broken to $\mathcal{N}$=1 by a superpotential $W = \sum_k g_k \mathrm{Tr}\Phi^k$ for adjoint chiral multiplet Φ, as well as for other gauge groups [2], with the addition of matter [3, 4], with multi-trace superpotentials [5] and for deformations of $\mathcal{N}$=4 gauge theories [6]. The types of results that can be derived or verified from the matrix model dual include effective glueball superpotentials [1], Seiberg duality [7], and the Affleck-Dine-Seiberg superpotential at low energies [3, 8]. Several techniques have been developed along these lines. The low energy superpotential after integrating out all of the matter can be derived in some cases, including nonperturbative effects, from just planar diagrams in the matrix model [1]. A new type of duality emerged by relating weakly coupled theories upon varying the tree level couplings so the theory passes through strong coupling [9]. Similar results have been derived through consistency with the Konishi anomaly [10], and these results are also expected to be consistent with the picture of monopole condensation as described by Seiberg and Witten [11]–[14]. It is the last of these points of view that we consider in this paper. We will use the matrix model solutions to relate vacua of different gauge theories.

In some sense this is a type of duality, although it follows naturally from an $\mathcal{N}=2$ non-renormalization theorem which is inherited by the $\mathcal{N}=1$ theory obtained by deforming the $\mathcal{N}=2$ theory. We will also study the matrix model solution for the $N_f=N_c$ theory in detail and demonstrate explicitly that the location in moduli space where monopole condensation takes place is consistent with the quantum modified constraint and factorization of Seiberg-Witten curves in *all* vacua of the theory, despite the complications due to the presence of baryons.

There is a classic picture of confinement as arising from the dual Meissner effect associated with the condensation of monopoles [15]. Just as superconductors confine magnetic fields when electrons condense, gauge theories are expected to confine electric fields when magnetic monopoles condense. While there is some evidence for this picture in ordinary QCD on the lattice [16], the occurrence of monopole condensation with confinement is better understood in certain supersymmetric theories [11].

In particular, consider $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(N_c)$ Yang-Mills theory with N_f fundamental massless flavors. The vacuum structure of these theories was studied in detail in [18]. The moduli space of vacua is split into Higgs branches, in which the matter scalars q_i have vacuum expectation values (vevs), and the Coulomb branch, in which the adjoint scalars ϕ^a, $a=1,\ldots,N_c^2-1$ have vevs. At generic points in the moduli space q_i and ϕ^a do not both have vevs together. The exception is along special submanifolds in the moduli space, called roots, where the Higgs and Coulomb branches meet. The roots are themselves classified by the gauge symmetry in the vacuum. There are non-baryonic roots in which the unbroken gauge symmetry is $\mathrm{SU}(r)\times\mathrm{U}(1)^{N-r}$ with $r\leq[N_f/2]$, and if $N_f\geq N_c$ there is also a baryonic root in which the gauge symmetry is $\mathrm{SU}(N_f-N_c)\times\mathrm{U}(1)^{2N_c-N_f}$. (If $N_f{=}N_c$ then the unbroken gauge symmetry at the baryonic root is just $\mathrm{U}(1)^{N_c-1}$.) At generic points on the Coulomb branch the unbroken gauge symmetry is $\mathrm{U}(1)^{N_c-1}$, so the roots typically have an enhanced gauge symmetry if the quarks are massless.

Furthermore, the roots contain special points or submanifolds on which monopoles or dyons (which we will refer to collectively as monopoles) become massless. As we will review, upon breaking to $\mathcal{N}{=}1$ supersymmetry the only vacua that survive are those points on the roots of the moduli space in which each of the unbroken U(1)'s has monopoles charged under it which then condense giving rise to confinement of electric charges as discussed above. These points typically occur at strong coupling, and the weak coupling degrees of freedom, $\Phi\equiv\sum_{a=1}^{N_c^2-1}\Phi^a\,T^a$ with T^a generators of $\mathrm{SU}(N_c)$ in the fundamental representation, are not a good description of the theory near these points. Instead, there are weakly coupled dual degrees of freedom, $\Phi_D\equiv\sum_{a=1}^{N_c^2-1}\Phi_D^a\,T^a$, which are defined at each point in the moduli space up to gauge equivalences.

The D-term equations of motion $[\Phi,\Phi^\dagger]=0$ force the adjoint to belong to the Cartan subalgebra of $\mathrm{SU}(N_c)$ so that the adjoint scalars ϕ_D can be diagonalized, and the N_c-1 independent diagonal elements form a basis of adjoints of the unbroken $\mathrm{U}(1)^{N_c-1}$ gauge group along the Coulomb branch. We will concentrate on the theory with $N_f=N_c$, and we will be most interested in those roots of the Higgs branches at which monopoles Q^i_{mon} and/or quarks Q_i become massless. At the baryonic root N_c-1 of these monopoles are each charged under one of the unbroken U(1)'s, and one of the monopoles, Q^N_{mon}, has the opposite charge under each of the U(1)'s. The monopoles form $\mathcal{N}=2$ hypermultiplets, so for each $\mathcal{N}$=1 chiral multiplet Q^i_{mon} there is a charge conjugated chiral multiplet $\tilde{Q}^i_{mon}$. It is convenient to collect the monopoles into a fundamental of the high energy SU(N) gauge group, Q_{mon} and $\tilde{Q}_{mon}$. The monopoles acquire the $\mathcal{N}=2$ superpotential,

$$W_{\rm mon} = {\rm Tr}\, Q_{\rm mon}\, \Phi_D\, \widetilde{Q}_{\rm mon}\,, \tag{1.1}$$

where the trace is over the SU(N_c) gauge group.[1] The monopoles are massless when the vev of Φ_D vanishes.

The adjoint scalars ϕ (or ϕ_D) are not gauge invariant: ϕ lies in the Cartan subalgebra of SU(N_c), and there are elements of SU$(N_c)/\,$U$(1)^{N_c-1}$ which leave ϕ in the Cartan subalgebra but permute the diagonalized elements of ϕ. Instead, it is convenient to parametrize the Coulomb branch by gauge invariant polynomials [17],

$$u_k = \frac{1}{k}\,{\rm Tr}\,\Phi^k\,, \qquad k = 2, \ldots, N\,. \tag{1.2}$$

If $\mathcal{N}=2$ supersymmetric mass terms of the form $\sum_{i=1}^{N_f} m_i {\rm Tr}\, Q_i \tilde{Q}_i$ are added to the superpotential then generically the Higgs branch will be completely lifted. From the point of view of the classical theory the reason is that the combination of D-term and adjoint F-term equations of motion force the squark vevs, written as $N_c \times N_f$ matrices (with $N_f = N_c$ here) to take either the non-baryonic form [18]:

$$Q = \begin{pmatrix} q_1 & & & & & \\ & \ddots & & & & \\ & & q_{[N_f/2]} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \qquad \tilde{Q}^T = \begin{pmatrix} 0 & & & q_1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & q_{[N_f/2]} \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \tag{1.3}$$

or the baryonic form,

$$Q = \begin{pmatrix} q & & & & & \\ & q & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & q \end{pmatrix}, \qquad \tilde{Q}^T = \begin{pmatrix} \tilde{q} & & & & & \\ & \tilde{q} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \tilde{q} \end{pmatrix}. \tag{1.4}$$

In components the matter F-term equations are:

$$Q^i_a \Phi^a_b = m_i Q^i_b \tag{1.5}$$

$$\Phi^a_b \tilde{Q}^b_i = m_i \tilde{Q}^a_i\,. \tag{1.6}$$

[1] Note that the normalization of the superpotential W_{mon} differs from the usual $\mathcal{N}=2$ normalization by a factor of $1/\sqrt{2}$, but was chosen for consistency with reference [8, 13] which we refer to often in this paper.

With Φ diagonalized, it is clear that there are only nonbaryonic solutions with nonvanishing Q if pairs of masses are equal, and there are only baryonic solutions if $\sum_i m_i = 0$ (for tracelessness of Φ). While we only considered the case $N_f = N_c$ above, the general result remains true for $N_f < N_c$: there are only non-baryonic Higgs branches if pairs of quark masses are equal. As we will see in section 5 this behavior is immediately apparent from the Type IIA and M-theory brane constructions of these theories.

If the $\mathcal{N} = 2$ theory is weakly broken to $\mathcal{N}=1$ SUSY with a superpotential of the form

$$W_{\mathcal{N}=1} = \sum_k g_k \, u_k \,, \tag{1.7}$$

then the effective superpotential including the light monopoles is $W_{\mathcal{N}=1} + W_{mon}$. At the baryonic branch in the $N_f = N_c$ theory there are no additional massless fields besides the monopoles.

Integrating out the u_k leads to,

$$\mathrm{Tr}\, Q_{\mathrm{mon}} \frac{\partial \Phi_D}{\partial u_k} \tilde{Q}_{\mathrm{mon}} = -g_k \,, \tag{1.8}$$

which implies that, as long as $\partial\Phi_D/\partial u_k$ is generic and nonvanishing, at least $N_c - 1$ of the N_c monopoles acquire vacuum expectation values. Then the equations of motion for the monopoles,

$$Q_{\mathrm{mon}} \, \Phi_D = \Phi_D \, \tilde{Q}_{\mathrm{mon}} = 0 \,, \tag{1.9}$$

late set,

$$\Phi_D(u_k) = 0 \,, \tag{1.10}$$

so the $\mathcal{N}=1$ vacua are stuck to the points at which the monopoles are massless, and they condense via (1.8). In the massless $\mathcal{N}=2$ theory these points are at the roots of the Higgs branches, but for generic masses such points continue to exist despite the absence of Higgs branches. We will continue to label vacua as baryonic or non-baryonic in a natural way in the matrix model solutions.

The low energy superpotential is then

$$W_{\mathrm{eff}} = W_{\mathcal{N}=1}(u_k^0) \,, \tag{1.11}$$

where $u_k^{(0)}$ are the solutions of (1.10).

If the quarks are massless, then at the non-baryonic roots the gauge symmetry is enhanced to $\mathrm{SU}(r) \times \mathrm{U}(1)^{N_c - r}$ with $r \leq N_f/2$. In addition to the massless monopoles there are N_f hypermultiplets in the fundamental of $\mathrm{SU}(r)$. If pairs of quark masses are equal but otherwise generic, then non-baryonic branches remain unlifted, but the $\mathrm{SU}(r)$ factor is broken to $\mathrm{U}(1)^{r-1}$ and there is one hypermultiplet charged under each of these $\mathrm{U}(1)$ factors. In any case, these light fields also acquire the $\mathcal{N} = 2$ superpotential $\sum_i Q_i \Phi \tilde{Q}_i$, and the Q_i equations of motion then set this term in the superpotential to zero as for the monopoles. The effective superpotential in the vacuum is then given by (1.11), just as in the baryonic vacuum.

If the adjoint mass (g_2 in the tree level superpotential) is large compared to the dynamical scale Λ, then the adjoint can be integrated out at high energies where the electric theory is weakly coupled. The description of the theory in terms of quarks is valid at that scale, and running down from there the mesons and baryons provide a valid low energy description of the theory. This description of the low energy theory should yield the same low energy superpotential as the monopole description (1.11), assuming there is no phase transition as the adjoint mass is varied. Including flavor masses, the tree level superpotential is,

$$W_{\text{tree}} = \sum_{i=1}^{N_f} \mathrm{Tr}\,(Q_i \Phi \tilde{Q}_i + m_i Q_i \tilde{Q}_i) + \sum_{k=2}^{N_c} g_k u_k\,, \tag{1.12}$$

where the first sum is the usual $\mathcal{N}=2$ superpotential and the second is the $\mathcal{N}=2$ breaking superpotential. In the $N_f{=}N_c$ theory there is a quantum modified constraint among the mesons and baryons. We will check that the matrix model prediction of the low energy superpotential with only g_2 nonvanishing agrees with the field theory prediction from (1.12) and the quantum modified constraint, and we will check in specific cases that these descriptions correctly predict the locations on the Coulomb branch at which monopole condensation occurs.

In section 2 we review the explicit matrix model results of Demasure and Janik [13] and we demonstrate that the extension of these results to the case $N_f = N_c$ correctly reproduces the nonperturbative field theory predictions for the low energy superpotential at the appropriate vacua, with the expected quantum modified constraints on the moduli. In section 3 we discuss the SU(2) and SU(3) theories in more detail, and demonstrate explicitly that the corresponding Seiberg-Witten curve factorizes as required for consistency with the picture of monopole condensation. In section 4 we use the matrix model solution to demonstrate a relation between vacua of theories with different numbers of colors and flavors. We elucidate some of these results via M5-brane constructions in section 5. We conclude in section 6.

2. Matrix model predictions for $N_f \leq N_c$

In this section we summarize the matrix model results of Demasure & Janik [13] for SU(N_c) with $N_f < N_c$ and we discuss the extension of these results to the case $N_f = N_c$.

2.1 $N_f < N_c$

Dijkgraaf and Vafa [1] suggested that $\mathcal{N}=2$ supersymmetric gauge theories perturbed by single-trace superpotentials of the form $W_{\mathcal{N}=1} = \sum_{k=2}^{N_c} g_k \,\mathrm{Tr}\,\Phi^k$ have an equivalent description in terms of a large-N matrix model whose action is simply

$$V(\Phi, Q_i, \tilde{Q}_i) = \frac{N}{S}\left(\mathrm{Tr}\, W_{\mathcal{N}=1}(\Phi) + \sum_i \left(m_i Q_i \tilde{Q}_i + Q_i \Phi \tilde{Q}_i\right)\right), \tag{2.1}$$

for the $N \times N$ matrix Φ, $1 \times N$ vector Q_i and $N \times 1$ vector $\tilde{Q}_i$. Another way to think of confinement below the scale of the flavor masses is via gaugino condensation in the $\mathcal{N}{=}1$ pure super Yang-Mills theory. Again, the various descriptions of the theory (in terms of quarks, monopoles or glueballs), are valid in different regimes of mass parameters or energies, and the assumption is that there is no phase transition upon varying these parameters so that the resulting low energy descriptions should be equivalent. In (2.1) S will be interpreted as the glueball superfield in the gauge theory. In particular, the leading N contributions to the matrix model partition function are interpreted in terms of the effective glueball superpotential in the gauge theory. If the glueball superpotential is written,

$$W_{\text{eff}}(S) = N_c \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} + \mathcal{F}_{\chi=1}\,, \tag{2.2}$$

then the matrix model prediction for $\mathcal{F}_{\chi=1}$ and $\mathcal{F}_{\chi=2}$ is,

$$e^{-\frac{N^2}{S^2}\mathcal{F}_{\chi=2}(S)-\frac{N}{S}\mathcal{F}_{\chi=1}(S)+\mathcal{O}(N^0)} = \int D\Phi\, DQ_i\, D\tilde{Q}_i\, e^{-\frac{N}{S}\left(\operatorname{Tr} W_{\mathcal{N}=1}(\Phi)+\sum_i m_i Q_i\tilde{Q}_i + Q_i\Phi\tilde{Q}_i\right)}. \tag{2.3}$$

The glueball is expected by symmetry arguments [19] to couple to $\log\Lambda^{2N_c-N_f}$ where Λ is the dynamical scale of the theory. Integrating in the glueball S via a Legendre transform of the low energy superpotential (1.11) with respect to $\log\Lambda^{2N_c-N_f}$ [20], the effective glueball superpotential derived from the matrix model should be equated with [13],

$$W_{\text{eff}}(S;\Omega,\Lambda) = S\log\left(\frac{\Lambda}{\Omega}\right)^{2N_c-N_f} + \sum_k g_k u_k^0(\Omega)\,. \tag{2.4}$$

In (2.4) the moduli are to be understood as functions of the quark masses and the scale Ω, which is set to the dynamical scale Λ by the S equation of motion. By integrating out the Ω auxiliary field, one obtains the low energy effective potential as a function of S. One can easily check that integrating out the glueball S reproduces the superpotential $W_{\mathcal{N}=1}(u_k^0)$.

By explicitly calculating the matrix model integrals in (2.3), determining the glueball superpotential (2.2) and comparing with (2.4), Demasure and Janik calculated the values of the moduli u_k^0 in the vacua of the theory as a function of the dynamical scale Λ and the flavor masses m_i. In fact, they were only interested in the baryonic vacua, so they selected a subset of the matrix model solutions. There is a $\mathbb{Z}_2^{N_f}$ ambiguity in the branch choice of a square root that appears in the identification of the effective superpotential in terms of matrix model variables. This can be seen either via the breakdown of the perturbative expansion of the matrix model free energies $\mathcal{F}_\chi$ at the branch point, or via the existence of multiple saddle point solutions in the matrix model expressed in terms of mesons as opposed to quarks [21]. By choosing the other branches we recover all of the vacua in the gauge theory. In the case that Higgs branches are unlifted there is a simple relationship between the various branches of the square roots and the Higgs branches.

Without repeating the details of the matrix model integrals, we quote the results in the form given in [13]. The result of the matrix model calculation for the moduli u_k in the $\mathcal{N}$=1 vacua is given implicitly in terms of the dynamical scale Λ of the high energy theory, and $u_1 = \operatorname{Tr}\,\Phi$, which is set to zero for the SU(N_c) theory. Generalizing the result of Demasure and Janik [13] to include the various branches of the matrix integrals (see also [21], we define the matrix model parameters T and R in terms of the field theory parameters $u_1 = \mathrm{Tr}\,\Phi$ and Λ via,

$$\begin{aligned}
u_1(R,T,m_i) &= N_c T - \sum_{i=1}^{n_-} \frac{1}{2}\left(m_i + T - \sqrt{(m_i+T)^2 - 4R}\right) - \\
&\quad - \sum_{i=n_-+1}^{N_f} \frac{1}{2}\left(m_i + T + \sqrt{(m_i+T)^2 - 4R}\right), \qquad (2.5)\\
\Lambda^{2N_c - N_f} &= R^{N_c - N_f} \prod_{i=1}^{n_-} \frac{1}{2}\left(m_i + T - \sqrt{(m_i+T)^2 - 4R}\right) \times \\
&\quad \times \prod_{i=n_-+1}^{N_f} \frac{1}{2}\left(m_i + T + \sqrt{(m_i+T)^2 - 4R}\right). \qquad (2.6)
\end{aligned}$$

Here, $n_- = 0, \ldots, N_f$ specifies the vacuum via the branch choice in the solution. (Taking n_- to $N_f - n_-$ leaves the choice of vacuum invariant.) The moduli in the $\mathcal{N}$=1 vacua are then given by [13]:

$$u_p^0 = N_c \mathcal{U}_p^{\mathrm{pure}}(R,T) + \sum_{i=1}^{n_-} \mathcal{U}_p^{-\,\mathrm{matter}}(R,T,m_i) + \sum_{i=n_-+1}^{N_f} \mathcal{U}_p^{+\,\mathrm{matter}}(R,T,m_i)\,, \qquad (2.7)$$

where

$$\begin{aligned}
\mathcal{U}_p^{\mathrm{pure}}(R,T) &= \frac{1}{p}\sum_{q=0}^{[p/2]} \binom{p}{2q}\binom{2q}{p} R^q T^{p-2q} \\
\mathcal{U}_{p\geq 2}^{+\,\mathrm{matter}}(R,T,m) &= \sum_{n=0}^{p-2} c_{p,n} R f_n(z^+) - \frac{v_p}{2}\left(m + T + \sqrt{(m+T)^2 - 4R}\right) \qquad (2.8)\\
\mathcal{U}_{p\geq 2}^{-\,\mathrm{matter}}(R,T,m) &= \sum_{n=0}^{p-2} c_{p,n} R f_n(z^-) - \frac{v_p}{2}\left(m + T - \sqrt{(m+T)^2 - 4R}\right).
\end{aligned}$$

$c_{p,n}$ and v_p are functions of R and T, and are given by [13],

$$\begin{aligned}
c_{p,n} &= 2^n R^{n/2} \sum_{k=0}^{\left[\frac{p-n-2}{2}\right]} \binom{2k}{k}\binom{p-1}{2k+n+1} R^k\, T^{p-n-2-2k} \qquad (2.9)\\
v_p &= \sum_{q=0}^{[p/2]} \frac{p-2q}{p}\binom{p}{2q}\binom{2q}{q} R^q\, T^{p-2q-1}. \qquad (2.10)
\end{aligned}$$

The $f_n(z^\pm)$ are polynomials determined in [13], up to the aforementioned branch choice which appears again in $z^\pm(R,T,m)$:

$$z^\pm = \frac{m+T}{2R}\left(m + T \mp \sqrt{(m+T)^2 - 4R}\right). \qquad (2.11)$$

The odd-looking choice of positive and negative roots in (2.11) was made for consistency with (2.8). The rth Higgs branch corresponds to all solutions with $N_f - |N_f - 2n_-| = r$, where n_- is the number of z^-'s in the solution (2.5)–(2.7). The reflection of all z^+'s and

z^-'s, i.e. taking $n_- \to N_f - n_-$, corresponds to solutions in the same rth Higgs branch so we will generically take $n_- \leq [N_f/2]$ and call $n_- = r$ with the understanding that we must also consider solutions with $n^- \to N_f - n_-$ in the same Higgs branch.

The choice $r = 0$ corresponds to the baryonic branch, and the other roots correspond to the non-baryonic branches. Recall also that we continue to label the vacua as baryonic or nonbaryonic depending on the choice of branches of the square roots, despite the fact that for generic masses they are unrelated to Higgs branches of the $\mathcal{N} = 2$ theory.

For simplicity we will concentrate on tree level superpotentials containing only u_2 and u_3, so from (2.8) we will only need $f_0(z)$ and $f_1(z)$, which are [13],

$$f_0(z) = \frac{1}{2(z-1)} \tag{2.12}$$

$$f_1(z) = \frac{3z-4}{6(z-1)^{3/2}}\,. \tag{2.13}$$

This completes the summary of matrix model results we will need, and for the complete discussion we refer the reader to [13].

2.2 $N_f = N_c$

Here we study the SU(N_c) theory with N_c flavors. While the matrix model approach has additional complications when baryons are present [4], we will see that the baryons vanish in all vacua except when the sum of the quark masses vanishes. The values of the moduli at the points where monopoles become massless vary smoothly with the quark masses. Hence, the matrix model predictions in previous sections for the moduli at the $\mathcal{N}$=1 vacua, which assumed that the baryons vanish, must correctly predict the moduli on the baryonic Higgs branches as well. This allows us to easily extend the results of [8, 13] to the case $N_f = N_c$. However, whenever additional massless degrees of freedom appear they must be included in the low energy description of the theory, so that the prediction of the low energy superpotential is expected to be modified. In fact, it has been argued that the quantum modified constraint including the baryons ($\det X - B\tilde{B} = \Lambda_{low}^{2N}$) follows from the matrix model with sources for the baryons included in the matrix model [4]. We will check here that by assuming that the baryons vanish, the matrix model predictions for the moduli u_k^0 are consistent with a field theory analysis, and in section 3 we will demonstrate that these results correctly predict factorization conditions for related Seiberg-Witten curves.

As in the theories with $N_f < N_c$ studied in [13], $z^\pm(R, T, m_i)$ will be related to the nonvanishing components of the diagonalized $N_f \times N_f$ meson matrix, $X_{ij} = M\tilde{X}_i\delta_{ij}$. More precisely,

$$\tilde{X}_i \equiv \frac{R}{m+T}\, z^\pm(R, T, m_i)\,, \tag{2.14}$$

where the sign choice is the same as that which determines the choice of vacuum in (2.5) and (2.6), for example. We immediately recognize (2.6) as the quantum modified constraint of the $\mathcal{N}$=1 theory with $N_f = N_c$ when the baryons vanish,

$$\det\, X_{ij} = \Lambda^{N_c} M^{N_c}\,, \tag{2.15}$$

where the one-loop matching condition determines the low energy dynamical scale below the mass M as $\Lambda_{low}^{2N_c} = \Lambda^{N_c} M^{N_c}$. We would like to demonstrate that the low energy effective superpotential at the various vacua agree with the matrix model prediction. Recall that the gauge theory at low energies has dual descriptions, either in terms of mesons and baryons or in terms of monopoles. We discussed the description in terms of monopoles in the Introduction. If the only $\mathcal{N}$=2 breaking term in the action is an adjoint mass, then the effective superpotential in the vacuum of the theory with $W_{\mathcal{N}=1} = M/2 \,\mathrm{Tr}\,\Phi^2 = M u_2$ is,

$$W_{\rm eff} = M\, u_2^0 \,, \tag{2.16}$$

where u_2^0 is evaluated at a solution of (1.10). We obtain the description in terms of mesons and baryons by taking the $\mathcal{N} = 2$ superpotential deformed by the superpotential (1.7), integrating out the adjoint and the massive matter, and adding the expected quantum modified constraint by hand. This is expected to be valid when the adjoint mass M is much larger than the dynamical scale Λ (so that the theory is weakly coupled at M), and then also for smaller values assuming there is no phase transition as the adjoint mass is varied. The full tree level superpotential is,

$$W = \frac{M}{2} \mathrm{Tr}\,\Phi^2 + \sum_{i=1}^{N_f} \mathrm{Tr}\, Q_i \Phi \tilde{Q}_i + \sum_{i=1}^{N_f} m_i\, Q_i \tilde{Q}_i + \lambda_\Phi\, \mathrm{Tr}\,\Phi \,, \tag{2.17}$$

where λ_Φ is a Lagrange multiplier enforcing the tracelessness of Φ in the SU(N_c) theory. The equations of motion for Φ are,

$$M \Phi_a^b + \sum_i Q_{ia} \tilde{Q}_i^b + \lambda_\Phi\, \delta_a^b = 0 \,, \tag{2.18}$$

the trace of which determines λ_Φ:

$$\lambda_\Phi = -\frac{1}{N} \sum_i \mathrm{Tr}\, Q_i \tilde{Q}_i \,. \tag{2.19}$$

Then we integrate out Φ using (2.18) to obtain,

$$W_{\rm eff} = -\frac{1}{2M} \mathrm{Tr}\, X^2 + \frac{1}{2N_c M} (\mathrm{Tr}\, X)^2 + \mathrm{Tr}\, mX \,, \tag{2.20}$$

where the traces are now over flavor indices, and we have defined,

$$X_{ij} \equiv \sum_{a=1}^{N_c} Q_{ia} \tilde{Q}_j^a \,, \tag{2.21}$$

$$m_{ij} \equiv m_i \delta_{ij} \,. \tag{2.22}$$

The quantum modified constraint can be added to (2.20) via a Lagrange multiplier, $\lambda_X \times (\det X - \Lambda^{N_c} M^{N_c})/M^{2N_c - 1}$. The vacua are determined by the equations of motion that follow from (2.20) together with the quantum modified constraint, which are solved by a diagonal meson matrix, $X_{ij} = M\tilde{X}_i\delta_{ij}$. In order to compare the results with the Seiberg-Witten description of the theory it is convenient to define $\tilde{X}_{ij} \equiv X_{ij}/M$. Then (2.20) becomes,

$$W_{\text{eff}} = M\left(-\frac{1}{2}\text{Tr}\,\tilde{X}^2 + \frac{1}{2N_c}(\text{Tr}\,\tilde{X})^2 + \text{Tr}\,m\tilde{X} + \lambda_X(\det\tilde{X} - \Lambda^{N_c})\right), \tag{2.23}$$

and the equations of motion are,

$$-\tilde{X}_i + \frac{1}{N}(\text{Tr}\,\tilde{X}) + m_i + \frac{\lambda_X}{\tilde{X}_i}\prod_{k=1}^{N_f}\tilde{X}_k\,. \tag{2.24}$$

Note that we have implicitly assumed that the baryons vanish in the analysis above. Including the possibility for baryonic vevs, the Lagrange multiplier term in (2.23) is modified to $\lambda_X(\det\,\tilde{X} - B\bar{B} - \Lambda^{Nc})$, where with the inclusion of the Lagrange multiplier we are to treat B and $\bar{B}$ as independent of the mesons $\tilde{X}$. Then the B and $\bar{B}$ equations of motion are:

$$\lambda_X B = \lambda_X\bar{B} = 0\,. \tag{2.25}$$

Hence, the baryons can only be nonvanishing if the Lagrange multiplier λ_X vanishes. So let us assume that $\lambda_X = 0$. Then, summing the equations of motion (2.24) over the N_c flavors we easily see that a solution exists only if $\sum_i m_i = 0$. As explained earlier, the fact that such solutions occur only for isolated regions in the parameter space of the theory, in addition to the fact that the moduli vary smoothly with these parameters, implies that the moduli are independent of the baryon vevs in the baryonic vacua. We will see an example of this in section 3.1.

We will now demonstrate that the alternative description for the low energy superpotential as $W_{\text{eff}} = Mu_2^0$, with the matrix model prediction for u_2 in the vacuum, agrees with (2.23) and (2.24). From (2.7) and (2.8), we have,

$$\begin{aligned} u_2^0(R,T,m_i) = {} & N_c\left(\frac{T^2}{2} + R\right) + \\ & + \frac{1}{4}\sum_{i=1}^{r}\left[(m_i - T)\left(m_i + T - \sqrt{(m_i+T)^2 - 4R}\right) - 2R\right] + \\ & + \frac{1}{4}\sum_{i=r+1}^{N_f}\left[(m_i - T)\left(m_i + T + \sqrt{(m_i+T)^2 - 4R}\right) - 2R\right]. \end{aligned} \tag{2.26}$$

We can determine T using (2.5) with u_1=0 and the identification (2.14), from which it follows that,

$$T = \frac{1}{N_c}\,\text{Tr}\,\tilde{X}\,. \tag{2.27}$$

Then, solving for R from (2.14) and tracing over the flavor indices we obtain,

$$R = \frac{1}{N_c}\left(-\text{Tr}\,\tilde{X}^2 + \frac{1}{N_c}(\text{Tr}\,\tilde{X})^2 + \text{Tr}\,m\tilde{X}\right). \tag{2.28}$$

Substituting these relations for the matrix model variables R and T into (2.26) gives the matrix model prediction for the effective superpotential $W_{\rm eff} = Mu_2^0$ in terms of the mesons $\tilde{X}_{ij}$ at the vacua:

$$W_{\rm eff}^{\rm vac} = M\left(-\frac{1}{2}\,{\rm Tr}\,\tilde{X}^2 + \frac{1}{2N_c}({\rm Tr}\ \tilde{X})^2 + {\rm Tr}\ m\tilde{X}\right), \tag{2.29}$$

which, together with the quantum modified constraint (2.15), exactly matches the field theory result (2.23). It remains to derive the equations of motion (2.24) from the matrix model.

Solving for R from the definition of $\tilde{X}_i$ in the matrix model (2.14) we have,

$$R = -\tilde{X}_i^2 + (m_i + T)\tilde{X}_i\,. \tag{2.30}$$

Recalling (2.27), if $\tilde{X}_i \neq 0$ (2.30) can be rewritten,

$$-\tilde{X}_i + \frac{1}{N_c}\sum_i \tilde{X}_i + m_i - \frac{R}{\tilde{X}_i}\,. \tag{2.31}$$

Comparing with the equations of motion (2.24), we see that the matrix model predicts the expected equations of motion and identifies the Lagrange multiplier with,

$$\lambda_X = -\frac{R}{\prod_k \tilde{X}_k} = -\frac{R}{\Lambda^{N_c}}\,, \tag{2.32}$$

where in the last step we used the quantum modified constraint (assuming the baryons vanish), which we have already seen follows from the matrix model.

This completes the demonstration that the matrix model prediction for the low energy superpotential in the $N_f = N_c$ theory agrees with the field theory. However, we would like to stress that the matrix model does more. If the two equations (2.5) and (2.6) defining the matrix model parameters R and T can be solved, then the values of all of the moduli u_p^0 are determined by the matrix model in terms of those two parameters. This follows from an analysis similar to that above with a generic superpotential $\sum_k g_k\,{\rm Tr}\,\Phi^k$ in place of the mass term $M/2\,{\rm Tr}\,\Phi^2$. In the following we explicitly check in certain cases that the above solutions factorize the appropriate Seiberg-Witten curves.

3. Factorization of Seiberg-Witten curves

The gauge theories discussed above would have $\mathcal{N} = 2$ supersymmetry were it not for the tree level superpotential $W_{\mathcal{N}=1}$. The low energy dynamics of these theories on the Coulomb branch has a solution in terms of a Seiberg-Witten curve [11, 12]. The $\mathcal{N} = 2$ prepotential of the low energy effective theory on the Coulomb branch is determined in terms of integrals of certain one-forms over the cycles of the Seiberg-Witten curve. In addition, the spectrum of monopoles and dyons is also determined by such integrals. As discussed earlier, when the theory is weakly broken to $\mathcal{N} = 1$ supersymmetry by the tree level superpotential the Coulomb branch is lifted except for those points at which monopoles or quarks would be

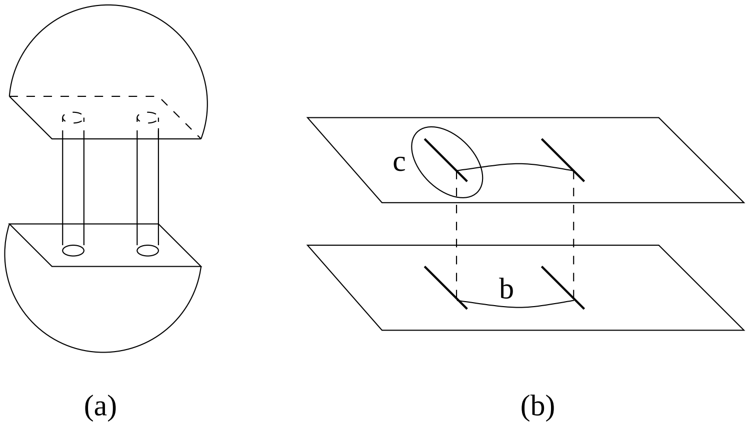

Figure 1: (a) Double sheeted cover of x-plane with branch cuts as a torus. (b) c-type and b-type cycles on the cut x-plane.

massless, and the monopoles condense. There is not expected to be a phase transition as the tree level couplings are varied, so the monopoles are expected to condense also when the theory is strongly broken to $\mathcal{N}=1$. This idea is tested by the matrix model description of the theory. In particular, the matrix model predictions for the moduli at the ground states of the theory are supposed to correspond to the points at which the monopoles of the $\mathcal{N}=2$ theory would be massless.

The Seiberg-Witten curves for these theories are hyperelliptic, taking the form,

$$y^2 = F_{2N_c}(x)\,, \tag{3.1}$$

where $F_{2N_c}(x)$ is a polynomial of order $2N_c$ in x, and is also a function of the moduli u_k, or equivalently vevs of the symmetric polynomials, $s_k = (-1)^k \sum_{i_1<\cdots<i_k} \phi_{i_1}\cdots\phi_{i_k}$, $k=2,\ldots,N$. For SU(N_c) with N_c flavors, the curve takes the form [22][2]

$$y^2 = \left(\sum_{k=0}^{N_c} s_k\, x^{N_c-k} + \Lambda^{N_c}\right)^2 - 4\Lambda^{N_c}\prod_{j=1}^{N_c}(x+m_j)\,. \tag{3.2}$$

The complex variable y is defined on a double-sheeted cover of the x-plane. There are branch points at the roots of $F_{2N_c}(x)$ which are joined in pairs to form cuts. Identifying the point at infinity, the hyperelliptic curve has the topology of a genus N_c-1 surface, as illustrated in figure 1a. There are correspondingly two sets of one-cycles on the Seiberg-Witten curve, as illustrated in figure 1b on the x-plane. These cycles are labeled c_i and b_i, where $i=1,\ldots,N-1$. For each pair of cycles there is a holomorphic one-form ω_i which is defined globally on the curve. For a hyperelliptic curve (3.1), a basis of the one-forms can be written as [23],

$$\omega_i = \frac{dx\, x^{N_c-i}}{y(x)}\,, \qquad i=2,\ldots,N_c\,. \tag{3.3}$$

As mentioned earlier, the gauge theory at strong coupling has a weakly coupled description in terms of a dual adjoint field Φ_D. The dual field Φ_D can be thought of as a particular

[2]Our convention for the dynamical scale is different than that of [22], namely, $\Lambda^{2N_c-N_f} = 4\Lambda_{HO}^{2N_c-N_f}$.

combination of the "electric" field Φ and the "magnetic" field Φ_d. In terms of the vacuum expectation values a^i and a_d^i of the diagonalized fields Φ and Φ_d the spectrum of dyons with electric and magnetic charges n_e^i and n_m^i (and not carrying global U(1) charges broken by the hypermultiplet masses m_i) is given by,

$$M_{nm} = \left| \sum_i (n_e^i a^i + n_m^i a_d^i) \right| . \tag{3.4}$$

The vevs a_i and a_d^i are determined as functions of the moduli up to gauge transformations via the identification [24, 22],

$$\frac{\partial a^i}{\partial s_k} = \int_{c_i} \omega_k \tag{3.5}$$

$$\frac{\partial a_d^i}{\partial s_k} = \int_{b_i} \omega_k \, . \tag{3.6}$$

Integrating these equations with respect to s_k and using the weak coupling vevs determines a^i and a^k. Equivalently, one can define a meromorphic one-form (holomorphic up to a simple pole), the Seiberg-Witten one-form λ_{SW}, such that the vevs are given by [24, 22],

$$a^i = \int_{c_i} \lambda_{\rm SW} \tag{3.7}$$

$$a_d^i = \int_{b_i} \lambda_{\rm SW} \, . \tag{3.8}$$

The dyon masses (3.4) are then,

$$M_{nm} = \left| \int_{n_e c_i + n_m b_i} \lambda_{\rm SW} \right| . \tag{3.9}$$

A dyon becomes massless when a cycle of the Seiberg-Witten curve vanishes. (A similar story applies when quarks become massless, but the masses (3.4) contain a term proportional to the mass of the quark and the global U(1) charge broken by the quark mass [12].) This happens when two of the roots of $F_{2N_c}(x)$ coincide, as is evident from figure 1b. When two roots collide either a branch cut shrinks to zero size or two branch cuts collide. Either way, a cycle vanishes. The degrees of freedom which become massless at a given singularity are determined by the beta function of the weakly coupled theory. The beta function is calculated by studying how the period matrix,

$$\tau_{ij} \equiv \frac{\partial a_d^i}{\partial a^j} \, , \tag{3.10}$$

determined by (3.7) and (3.8), transforms when the moduli make a loop around the singularity. This monodromy depends on the beta function because according to the Seiberg-Witten analysis τ_{ij} is identified with the (running) gauge coupling function which appears in the action on the Coulomb branch,

$$S \sim {\rm Im} \int d^4x \int d^2\theta \, \tau_{ij} \, W^{\alpha\, i} W_\alpha^j \, , \tag{3.11}$$

with $W^{\alpha i}$ the field strength chiral superfield for the ith U(1) gauge group factor. If several quarks (under an unbroken SU(r) factor in the gauge symmetry at a non-baryonic root) or mutually local monopoles (charged under orthogonal U(1) gauge group factors) become massless, the roots of $F_{2N_c}(x)$ will coincide in pairs, with one pair for each U(1) factor under which degrees of freedom become massless. Higher order roots indicate additional massless states, as in the presence of mutually nonlocal monopoles at Argyres-Douglas points [25] or an enhanced gauge symmetry at a non-baryonic root in the theory with specially tuned quark masses.

At the $\mathcal{N}$=1 vacua discussed in the previous sections $N_c - 1$ mutually local monopoles become massless and condense at the baryonic branch, and on the non-baryonic branches $N_c - r - 1$ monopoles (charged under $N_c - r - 1$ unbroken U(1) factors) condense and r quarks become massless under the remaining unbroken subgroup of SU(r). In either case, in the corresponding $\mathcal{N} = 2$ theory $N_c - 1$ degrees of freedom become massless, so the Seiberg-Witten curve is expected to factorize as,

$$y^2 = f_2(x) \prod_{i=1}^{N_c-1} (x - x_i)^2 \,, \tag{3.12}$$

where $f_2(x)$ is a second order polynomial in x. So the matrix model provides a prediction for the values of the moduli s_k for which the polynomial $F_{2N_c}(x)$ factorizes as (3.12). This is in general a complicated algebraic problem. For the pure gauge theory (without matter) there is a known solution in terms of Chebyshev polynomials [26]. In certain limits factorization of Seiberg-Witten curves has been studied in theories with matter [27, 28]. Only recently has a more general study been done via the matrix model duals of the broken $\mathcal{N} = 2$ gauge theories [13, 8], as reviewed in section 2 for the SU(N_c) gauge theory with $N_f < N_c$, and also as extended to the case N_f=N_c there. The explicit predictions for curve factorization have been tested in limited cases [13, 14]. In general the algebraic problem of factorization is quite complicated and has evaded an analytical solution. In this section we study the results for SU(2) with 2 flavors and SU(3) with 3 flavors in more detail, and we explicitly test factorization of the Seiberg-Witten curves in that case. In the case that quark masses are chosen to be pairwise equal we will be able to classify the branches of moduli space in a simple way, as will be apparent from the M5-brane construction.

3.1 SU(2) with 2 flavors

For simplicity we will take $m_1 = m_2 \equiv m$. In this case there is a non-baryonic Higgs branch in the $\mathcal{N} = 2$ theory. After breaking to $\mathcal{N}$=1 with the adjoint mass the quantum modified constraint (2.15) on the diagonalized meson vevs is,

$$\tilde{X}_1 \tilde{X}_2 = \Lambda^2 \,. \tag{3.13}$$

Then the effective superpotential (2.29) can be written in terms of $\tilde{X}_1 \equiv \tilde{X}$ as,

$$\frac{W_{\text{eff}}(\tilde{X})}{M} = -\frac{1}{2}\left(\tilde{X}^2 + \frac{\Lambda^4}{\tilde{X}^2}\right) + \frac{1}{4}\left(\tilde{X} + \frac{\Lambda^2}{\tilde{X}}\right)^2 + m\left(\tilde{X} + \frac{\Lambda^2}{\tilde{X}}\right). \tag{3.14}$$

The stationary points of $W_{\rm eff}$ are at the meson vevs:

$$\tilde{X} = \pm\Lambda,\ m \pm \sqrt{m^2 - \Lambda^2}\,. \tag{3.15}$$

These solutions give rise to the following values of the effective superpotential at the vacua:

$$\frac{W_{\rm eff}^{\rm vac}}{M} = u_2^0 = \pm 2m\Lambda \text{ or } \Lambda^2 + m^2\,. \tag{3.16}$$

The last solution above is doubly degenerate (corresponding to exchanging $\tilde{X}_1$ and $\tilde{X}_2$) and hence there are four vacua for this theory. These solutions are the matrix model and field theory prediction for u_2 at the vacua of the $\mathcal{N}=2$ theory broken to $\mathcal{N}$=1 by an adjoint mass.

One can also obtain the effective superpotential directly from matrix model. The baryonic ($r=0$) solutions for R and T are given by,

$$2T = \left(m + T - \sqrt{(m+T)^2 - 4R}\right) \tag{3.17}$$

$$\Lambda^2 = \left(\frac{1}{2}(m + T - \sqrt{(m+T)^2 - 4R})\right)^2\,, \tag{3.18}$$

which has the solutions $R = -\Lambda m$, $T = -\Lambda$ and $R = \Lambda m$, $T = \Lambda$. Substituting these values into (2.26) produces the first set of effective potentials in (3.16). The equation of motion in the non-baryonic ($r=1$) branch are,

$$2T = (m+T) \tag{3.19}$$

$$\Lambda^2 = R \tag{3.20}$$

which yields the second set of solutions in (3.16).

For SU(2) the Seiberg-Witten curve (3.2) is,

$$y^2 = \left(x^2 - u_2 + \Lambda^2\right)^2 - 4\Lambda^2(x+m_1)(x+m_2)\,. \tag{3.21}$$

When the Seiberg-Witten curve factorizes it takes the form,

$$y^2 = (x-e)^2(x^2 + bx + c)\,. \tag{3.22}$$

We equate (3.21) and (3.22) to solve for the factorized form of the curve and the value of u_2 at which the curve factorizes. For $m_1 = m_2 = m$ the solutions are,

$$\begin{aligned} u_2 &= m^2 + \Lambda^2\,, \qquad y^2 = (x+m)^2((x-m)^2 - 4\Lambda^2)\,, \qquad \text{or} \\ u_2 &= \pm 2m\Lambda \qquad y^2 = (x \pm \Lambda)^2\left(x^2 - 2\Lambda\, x + \Lambda^2 \mp 4m\Lambda\right)\,. \end{aligned} \tag{3.23}$$

These values for u_2 are in agreement with the matrix model prediction (3.16). The first solution in (3.23) corresponds to the non-baryonic root, and the second to the baryonic root.

We also checked agreement between the various descriptions in the case $m_1 = -m_2 = m$, for which there is a baryonic Higgs branch in the $\mathcal{N}=2$ theory. Allowing for a baryon vev, the effective superpotential after using the quantum modified constraint is,

$$\frac{W_{\rm eff}(\tilde X)}{M} = -\frac{1}{2}\left(\tilde X^2 + \frac{(\Lambda^2+B\bar B)^2}{\tilde X^2}\right) + \frac{1}{4}\left(\tilde X + \frac{\Lambda^2+B\bar B}{\tilde X}\right)^2 + m\left(\tilde X + \frac{\Lambda^2+B\bar B}{\tilde X}\right), \tag{3.24}$$

where B and $\bar B$ have been rescaled by the adjoint mass M as $\tilde X$ was. Solving the equations of motion for $\tilde X$ we find that at the non-baryonic vacua (for which $B=\bar B=0$), $W_{\rm eff} = \Lambda^2 \pm 2im\Lambda$, and at the baryonic roots we find $W_{\rm eff} = m^2$, independent of B and $\bar B$.

3.2 SU(3) with 3 flavors

The quantum modified constraint (2.15) on the diagonalized meson vevs is,

$$\tilde X_1 \tilde X_2 \tilde X_3 = \Lambda^3\,. \tag{3.25}$$

Then the effective superpotential (2.29) can be written in terms of mesons and a Lagrange multiplier λ_X as,

$$\frac{W_{\rm eff}(\tilde X)}{M} = -\frac{1}{2}\,{\rm Tr}(\tilde X^2) + \frac{1}{6}({\rm Tr}\,\tilde X)^2 + {\rm Tr}\, m\tilde X + \lambda_X(\det\,\tilde X - \Lambda^3)\,. \tag{3.26}$$

At the baryonic solutions the meson vevs are all equal. The vevs are determined by the stationary points of $W_{\rm eff}$ and are found to be,

$$\tilde X_k = \Lambda\omega^k \qquad k=1,2,3\,, \tag{3.27}$$

where $\omega = e^{2\pi i/3}$ is a third root of unity. At the remaining (non-baryonic) solutions two of three meson vevs are equal, namely,

$$\tilde X_1 = \tilde X_2 = m - \frac{m^2}{\Delta} - \Delta, \qquad \tilde X_3 = \frac{3m - \tilde X_1}{2} = m + \frac{m^2}{2\Delta} + \frac{\Delta}{2} \tag{3.28}$$

$$\tilde X_1 = \tilde X_2 = m - \omega^2\frac{m^2}{\Delta} - \omega\Delta, \qquad \tilde X_3 = m + \omega^2\frac{m^2}{2\Delta} + \frac{\omega\Delta}{2} \tag{3.29}$$

$$\tilde X_1 = \tilde X_2 = m - \omega\frac{m^2}{\Delta} - \omega^2\Delta, \qquad \tilde X_3 = m + \omega\frac{m^2}{2\Delta} + \frac{\omega^2\Delta}{2}\,, \tag{3.30}$$

where

$$\Delta = \left(\Lambda^3 - m^3 + \sqrt{\Lambda^6 - 2\Lambda^3 m^3}\right)^{1/3}\,. \tag{3.31}$$

The first set of solutions give rise to the following values of the effective superpotential at the three vacua:

$$\frac{W_{\rm eff}^{\rm vac}}{M} = u_2 = 3m(\Lambda^3)^{1/3}\,. \tag{3.32}$$

The second set of solutions give rise to three vacua as well. The first solution of the second set leads to superpotential,

$$\frac{W_{\rm eff}^{\rm vac}}{M} = u_2 = \frac{3}{4}(m^2 + 4m\tilde X_1 - \tilde X_1^2) = \frac{3}{4}\left(2m^2 - \frac{m^4}{\Delta^2} - \frac{2m^3}{\Delta} - 2m\Delta - \Delta^2\right)\,. \tag{3.33}$$

The superpotential at the other two vacua can be obtained by replacing Δ with $\omega\Delta$ and $\omega^2\Delta$. Notice that the first solution is invariant under $\Delta \leftrightarrow \frac{m^2}{\Delta}$. In fact, each of these three solutions is three-fold degenerate (corresponding to permutations of the three meson vevs). Thus, there are a total of twelve vacua for this theory.

Similarly, the effective superpotential can be obtained from matrix model. The equations determining R and T in the baryonic ($r = 0$) vacua are,

$$3T = \frac{3}{2}(m + T - \sqrt{(m+T)^2 - 4R}) \tag{3.34}$$

$$\Lambda^3 = \left(\frac{1}{2}(m + T - \sqrt{(m+T)^2 - 4R})\right)^3, \tag{3.35}$$

which have the solutions $R = \Lambda m$, $T = \Lambda$, $R = \Lambda\omega m$, $T = \Lambda\omega$ and $R = \Lambda\omega^2 m$, $T = \Lambda\omega^2$. Substituting these values into (2.26) produce the first set of effective potential in (3.32). One can also obtain the moduli u_3 and in this branch, it is given by

$$u_3 = 3m(\Lambda\omega^k)^2 \qquad k = 1, \ldots 3. \tag{3.36}$$

The R and T equations for the ($r = 1$) vacua are,

$$3T = (m + T) + \frac{1}{2}(m + T - \sqrt{(m+T)^2 - 4R}) \tag{3.37}$$

$$\Lambda^3 = R\left(\frac{1}{2}(m + T - \sqrt{(m+T)^2 - 4R})\right), \tag{3.38}$$

which has the solutions,

$$T = m - \frac{m^2}{2\Delta} - \frac{\Delta}{2} \tag{3.39}$$

$$R = -\frac{1}{2}\left(\frac{m^4}{\Delta^2} + \frac{m^3}{\Delta} + m\Delta + \Delta^2\right). \tag{3.40}$$

Substituting to (2.26), we find the superpotential given in (3.33). By replacing Δ with $\Delta\omega$ and $\Delta\omega^2$, we obtain the other two vacua. The modulus u_3 is given in this vacuum by,

$$u_3 = \Lambda^3 + m(u_2 - m^2), \tag{3.41}$$

with u_2 as given by (3.33).

For SU(3) with three flavors of equal mass, the Seiberg-Witten curve (3.2) can be written as,

$$y^2 = \left(x^3 - u_2 x - u_3 + \Lambda^3\right)^2 - 4\Lambda^3(x+m)^3. \tag{3.42}$$

The factorized curve takes the form,

$$y^2 = (x - e_1)^2(x - e_2)^2(x^2 + bx + c). \tag{3.43}$$

By matching (3.42) and (3.43), we find the value of u_2 and u_3 at which the curve factorizes. With u_2 and u_3 given by the baryonic solution (3.32), (3.36) as predicted by the field theory

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and matrix model, the curve indeed factorizes as,

$$y^2 = \left(x + \frac{1}{2}(\Lambda + \sqrt{\Lambda(-3\Lambda + 4m)}\right)^2 \left(x + \frac{1}{2}(\Lambda - \sqrt{\Lambda(-3\Lambda + 4m)}\right)^2 \times \\ \times \left(x^2 - 2\Lambda\, x + \Lambda^2 - 4m\Lambda\right). \tag{3.44}$$

At the non-baryonic solution given by (3.33) and (3.41), the Seiberg-Witten curve again factorizes as (3.43) with:

$$e_1 = -m \tag{3.45}$$

$$e_2 = \frac{\Delta}{2} + \frac{m^2}{2\Delta} \tag{3.46}$$

$$b = -2m + \Delta + \frac{m^2}{\Delta} \tag{3.47}$$

$$c = \frac{1}{e_2^2}\left((u_2 - m^2)^2 - 4\Lambda^3 m\right), \tag{3.48}$$

with Δ given by (3.31) and u_2 given by (3.33), in addition to solutions of the same form with $\Delta \to \Delta\omega$ and $\Delta \to \Delta\omega^2$.

As another consistency check on these solutions, consider the limit in which we keep $m\Lambda$ fixed while taking $\frac{m}{\Lambda}$ large. The mesons are heavy with respect to the strong scale of the theory. In this case, the low energy theory reduces to $\mathcal{N} = 1$ SU(N) SYM gauge theory with no matter. This theory has N distinct vacua located at $Mu_2 = N(\Lambda_{low}^{3N})^{1/N}$, with $\Lambda_{low}^3 = Mm\Lambda$ from the one-loop matching condition. A quick glance at the above examples for SU(2) and SU(3) gauge groups makes it clear that both the matrix model and curve factorization results are consistent with the pure gauge theory in the appropriate limit.

Before we leave this section we make an observation regarding some of these solutions that are indicative of a general result that we will prove in the following section. In the SU(2) theory with equal quark masses, the first factorized form of the Seiberg-Witten curve in eq. (3.23) has the form of $(x+m)^2$ times a polynomial whose interpretation is *a priori* unclear. In the SU(3) case with equal quark masses, the Seiberg-Witten curve factors similarly at the non-baryonic vacua. By insisting that the Seiberg-Witten curve factorize as,

$$y^2 = (x+m)^2 f_4(x, u_2, m),$$

we find

$$f_4(x, u_2, m) = (x^2 - m\,x - u_2 + m^2)^2 - 4\Lambda^3(x+m) \tag{3.49}$$

which describes the curve for the U(2) theory with $N_f = 1$, with U(1) adjoint vev $\tilde{u}_1 = m$ and $\tilde{u}_2 = u_2 - m^2/2$. This is not a coincidence, as we demonstrate in the following section.

4. Relations between vacua

In this section we test the decoupling of the Higgs and Coulomb branches, as predicted by an $\mathcal{N} = 2$ nonrenormalization theorem [18], via the matrix model solutions and Seiberg-Witten curves. We also discuss the persistence of $\mathcal{N} = 2$ nonrenormalization properties in the $\mathcal{N}$=1 theories discussed previously.

4.1 Relations between matrix model solutions

On non-baryonic Higgs branches of the $\mathcal{N} = 2$ SU(N_c) theory with N_f flavors, which we will call the (N_c, N_f) theory, some of the adjoint vevs ϕ_i become equal in magnitude to quark masses m_i. As mentioned earlier, the D-term and F-term equations of motion allow such solutions to exist only when pairs of quark flavors have equal masses. In the $\mathcal{N} = 2$ theory the squark vevs can be made arbitrarily large, and by the $\mathcal{N} = 2$ nonrenormalization theorem the squark vevs cannot influence the Coulomb branch of the remaining SU($N_c - r$) theory with $N_f - 2r$ flavors (if r pairs of flavors obtain nonbaryonic vevs in this way) [18]. In particular, the location on the Coulomb branch where monopoles become massless in the rth nonbaryonic branch of the $\mathcal{N} = 2$ (N_c, N_f) theory should be related to the effective superpotential at the vacua of the $\mathcal{N} = 1$ $(N_c - r, N_f - 2r)$ theory. The matrix model solutions allow us to test this picture.

For simplicity we will first consider the case when all quark masses vanish, and afterward discuss the massive theory. In the rth nonbaryonic branch we can rewrite equations (2.5) and (2.6) with $n_- = r$ as,

$$u_1(R,T) = \left(N_c - \frac{N_f}{2}\right) T - \left(\frac{N_f}{2} - r\right)\sqrt{T^2 - 4R}\,, \tag{4.1}$$

$$\Lambda^{2N_c - N_f} = R^{N_c - N_f + r}\left(\frac{1}{2}(T + \sqrt{T^2 - 4R})\right)^{N_f - 2r}\,. \tag{4.2}$$

Defining $\tilde{N}_c \equiv N_c - r$ and $\tilde{N}_f \equiv N_f - 2r$, we then have,

$$u_1(R,T) = \left(\tilde{N}_c - \frac{\tilde{N}_f}{2}\right) T - \frac{\tilde{N}_f}{2}\sqrt{T^2 - 4R}\,, \tag{4.3}$$

$$\Lambda^{2\tilde{N}_c - \tilde{N}_f} = R^{\tilde{N}_c - \tilde{N}_f}\left(\frac{1}{2}(T + \sqrt{T^2 - 4R})\right)^{\tilde{N}_f}\,. \tag{4.4}$$

These are the equations which define R and T in the baryonic solution of the SU($\tilde{N}_c$) theory with $\tilde{N}_f$ flavors.[3] We can now demonstrate that u_2^0 in the nonbaryonic (N_c, N_f) vacua with massless flavors equals u_2^0 in the vacua of the $(\tilde{N}_c, \tilde{N}_f)$ theory. From (2.26) we have,

$$\begin{aligned} u_2^0 &= \left(\frac{T^2}{2} + R\right)\left(N_c - \frac{N_f}{2}\right) + (N_f - 2r)(-T)\sqrt{T^2 - 4R} \\ &= \left(\frac{T^2}{2} + R\right)\left(\tilde{N}_c - \frac{\tilde{N}_f}{2}\right) + \tilde{N}_f(-T)\sqrt{T^2 - 4R}\,. \end{aligned} \tag{4.5}$$

Note that in the above story we could have exchanged all positive and negative square roots and the conclusion would have been the same. As we already mentioned, the rth set of nonbaryonic solutions corresponds to choosing either r positive or r negative square roots,

[3] In fact, with $u_1 \neq 0$ these equations also relate the vacua of the U(N_c) and U($\tilde{N}_c$) theories, as is also expected from the $\mathcal{N}$=2 nonrenormalization theorem.

and the baryonic vacua correspond to choosing either all positive or all negative square roots. In either case, we have shown that the values of u_2^0 agree in the corresponding baryonic and nonbaryonic solutions.

Alternatively, we could have interpreted the above observation as a relation between vacua on the $(r-n)$th non-baryonic branch of the (N_c, N_f) theory with the nth non-baryonic branch of the $(N_c-(r-n), N_f-2(r-n))$th theory. Hence, more generally, the value of the Coulomb branch modulus u_2 at the vacua in which monopoles are massless on the rth Higgs branch of the $\mathcal{N}=2$ (N_c, N_f) theory are equivalent to the effective superpotential at the vacua of the $\mathcal{N}=1$ (N_c-r, N_f-2r) theory, as promised.

This suggests that the Seiberg-Witten curve of the (N_c, N_f) theory at the r-th root factorizes in the following way (see also [27]):

$$y^2 = x^{2r}\left[P_{\tilde{N}_c}(x,\tilde{u}_k)^2 - 4\Lambda^{2\tilde{N}_c-\tilde{N}_f}x^{\tilde{N}_f}\right], \tag{4.6}$$

where $P_{\tilde{N}_c}(x,\tilde{u}_k) = \sum_{k=0}^{\tilde{N}_c}\tilde{s}_k x^{\tilde{N}_c-k}$. Note that

$$y^2 = P_{\tilde{N}_c}(x,\tilde{u}_k)^2 - 4\Lambda^{2\tilde{N}_c-\tilde{N}_f}x^{\tilde{N}_f}$$

is the Seiberg-Witten curve of the $(\tilde{N}_c, \tilde{N}_f)$ theory. (To compare with earlier literature, this factorization corresponds to the type I superconformal theory of [29, 27].[4]) Expanding (4.6),

$$\begin{aligned} x^{2r}\left[P_{\tilde{N}_c}(x,\tilde{u}_k)^2 - 4\Lambda^{2\tilde{N}_c-\tilde{N}_f}x^{\tilde{N}_f}\right] &= x^{2r}\left[(x^{\tilde{N}_c}-\tilde{u}_2x^{\tilde{N}_c-2}-u_3x^{\tilde{N}_c-3}+\cdots)^2-\Lambda^{2\tilde{N}_c-\tilde{N}_f}x^{\tilde{N}_f}\right] \\ &= x^{2r}x^{2\tilde{N}_c} - 2x^{2r}\tilde{u}_2x^{\tilde{N}_c-2}+\cdots \\ &= x^{2N_c} - 2\tilde{u}_2x^{2N_c-2}+\cdots . \end{aligned}$$

We find some evidence that the Seiberg-Witten curve of the (N_c, N_f) theory factorizes as (4.6) in the fact that $u_2 = \tilde{u}_2$, in accordance with the $\mathcal{N}=2$ nonrenormalization theorem. In fact, all of the $u_k = \tilde{u}_k$, $k = 2,\ldots,\tilde{N}_c$ in the curve factorized as (4.6), and the same should follow from the matrix model solutions although we have only checked this for u_2.

The story is only slightly more involved for the massive theory. On the rth nonbaryonic branch r elements of the diagonalized adjoint ϕ_i satisfy $\phi_i + m_i = 0$ for r of the quark masses m_i. Recall also that each of these m_i is the mass of a pair of quarks. In matching the moduli in the two theories as above, there are two additional differences to keep in mind between the massive and massless theories. First of all, in the (N_c, N_f) theory the adjoint is traceless. That implies that in the corresponding $(\tilde{N}_c, \tilde{N}_f)$ theory the trace of the adjoint vev, $\tilde{u}_1$, is determined by,

$$u_1 = \tilde{u}_1 - \sum_{i=1}^{r} m_i = 0 . \tag{4.7}$$

We can either think of this as turning on a vev for an extra U(1) in the $(\tilde{N}_c, \tilde{N}_f)$ theory, or as shifting the average hypermultiplet mass by $\sum_{i=1}^{r} m_i/\tilde{N}_f$.

[4]The $r = N_f/2$ case is an exception. It belongs to the class 3 or 4 SCFT, depending on whether $N_c - N_f/2$ is odd or even [27].

The other thing to keep in mind is that, in addition to the effects of the extra U(1) adjoint vev due to (2.5), $u_2 = 1/2\,\mathrm{Tr}\,\phi^2$ receives a contribution from the vevs $\phi_i = m_i$, so that,

$$u_2 = \tilde{u}_2 + \frac{1}{2}\sum_{i=1}^{r} m_i^2\,. \tag{4.8}$$

Keeping in mind these differences from the massless case, we can check that the moduli agree in the massive case just as for the massless case. Consider a vacuum on the rth non-baryonic branch of the $\mathcal{N}$=2 theory. After breaking to the $\mathcal{N}$=1 theory this corresponds to any of the vacua in the $(r+n)$th branch with $r+n \le [N_f/2]$. For the equations defining T and R we find,

$$\begin{aligned} T\left(\tilde{N}_c - \frac{1}{2}\tilde{N}_f\right) &= -\sum_{i=1}^{\tilde{N}_f-n}\frac{1}{2}\sqrt{(m_i+T)^2-4R} + \sum_{i=\tilde{N}_f-n+1}^{\tilde{N}_f}\frac{1}{2}\sqrt{(m_i+T)^2-4R} + \\ &\quad + \frac{1}{2}\sum_{i=1}^{\tilde{N}_f} m_i + \sum_{i=1}^{r} m_i\,, \qquad (4.9)\\ \Lambda^{2\tilde{N}_c-\tilde{N}_f} &= R^{\tilde{N}_c-Nt_f}\prod_{i=1}^{\tilde{N}_f-n}\frac{1}{2}\left(m_i+T-\sqrt{(m_i+T)^2-4R}\right)\times \\ &\quad \times \prod_{i=\tilde{N}_f-n+1}^{\tilde{N}_f}\frac{1}{2}\left(m_i+T+\sqrt{(m_i+T)^2-4R}\right), \qquad (4.10) \end{aligned}$$

where we have used the fact that r pairs of equal m_i appear in opposite branches of the square roots, corresponding to the rth non-baryonic branch of the $\mathcal{N}=2$ theory. Comparing with (2.5), we see that the last term in (4.9) plays the role of $\tilde{u}_1$, as expected. Then we can write (2.26) as,

$$\begin{aligned} u_2^0 &= \left(\frac{T^2}{2}+R\right)\left(\tilde{N}_c - \frac{\tilde{N}_f}{2}\right) - \sum_{i=1}^{\tilde{N}_f-n}\frac{1}{4}(m_i-T)\sqrt{(m_i+T)^2-4R} + \\ &\quad + \sum_{i=\tilde{N}_f-n+1}^{\tilde{N}_f}\frac{1}{4}(m_i-T)\sqrt{(m_i+T)^2-4R} + \frac{1}{2}\sum_{i=1}^{r} m_i^2 \\ &= \tilde{u}_2^0 + \frac{1}{2}\sum_{i=1}^{r} m_i^2\,, \qquad (4.11) \end{aligned}$$

with $\tilde{u}_2^0$ evaluated at the solutions of the $(\tilde{N}_c, \tilde{N}_f)$ theory as promised. As in the massless case, all positive and negative branches of the square roots could have been exchanged with the same conclusion.

Hence, in the generic case in which there exist pairs of quarks with equal mass, we have once again proved that the non-baryonic roots of the $\mathcal{N}=2$ (N_c, N_f) theory at which monopoles condense determine the low energy superpotential at all of the vacua of the $\mathcal{N}=1$ $(\tilde{N}_c, \tilde{N}_f)$ theory.

In accordance with this relation between the (N_c, N_f) and $(\tilde{N}_c, \tilde{N}_f)$ theories, the curve of (N_c, N_f) theory at the r-th root should factorize as

$$y^2 = (x+m_1)^2(x+m_2)^2\cdots(x+m_r)^2\left[P_{\tilde{N}_c}(x,\tilde{u}_k)^2 - 4\Lambda^{\tilde{N}_c}\prod_{k=r+1}^{N_f}(x+m_k)\right]. \qquad (4.12)$$

Again, $y^2 = P_{\tilde{N}_c}(x,\tilde{u}_k)^2 - 4\Lambda^{\tilde{N}_c}\prod_{k=r+1}^{N_f}(x+m_k)$ is the curve for $(\tilde{N}_c, \tilde{N}_f)$ theory with $\tilde{u}_1 = \sum_{i=1}^{r} m_i$. Now, as in the $m_i = 0$ case, the coefficient of x^{N_c-2} should be equal to $-2u_2$ for consistency with the picture developed above. Indeed, by expansion of eq. (4.12) and using eqs. (4.7) and (4.8), the coefficient of x^{N_c-2} is,

$$2(\tilde{u}_1^2 - \tilde{u}_2^2) - 4\tilde{u}_1\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} m_i^2 + 4\sum_{i<j} m_i m_j = -2u_2\,, \qquad (4.13)$$

as desired. We also note that the coefficient of x^{N_c-1} is $-\tilde{u}_1 + \sum_{i=1}^{r} m_i$, which vanishes by eq. (4.7).

These results highlight the power of the matrix model techniques and confirm the $\mathcal{N}=2$ nonrenormalization theorem in this context. We would like to stress the interesting fact that, because the Coulomb branch moduli at the $\mathcal{N}=1$ vacua are stuck to specific values determined by the $\mathcal{N}=2$ theory, the low energy superpotential of the $\mathcal{N}=1$ theory inherits the nonrenormalization properties of the $\mathcal{N}=2$ theory. That is why the decoupling of the Higgs and Coulomb branches of $\mathcal{N}=2$ theories has proven valuable for studying theories with only $\mathcal{N}=1$ supersymmetry.

4.2 Counting the solutions

The matrix model results can be used to reproduce the number of vacua of the $\mathcal{N}=1$ theory with equal mass $m_i = m$, computed in [27] (see also [21]). We present two different methods. First, we simply count the number of solutions of $u_1(R,T) = 0$ and eq. (2.6). The second method utilizes the relation between baryonic and non-baryonic branches discussed in the previous section, and provides another check of that relation.

First of all, $u_1 = 0$ implies

$$\sqrt{(m+T)^2 - 4R} = \frac{(2N_c - N_f)T - mN_f}{2r - N_f}\,. \qquad (4.14)$$

Assuming $r \neq N_f/2$, we use this to reduce eq. (2.6) to the following form:

$$\begin{aligned}\Lambda^{2N_c-N_f} = R^{N_c-N_f}&\left((m+T) - \frac{(2N_c-N_f)T - mN_f}{2r-N_f}\right)^r \times \\ &\times \left((m+T) + \frac{(2N_c-N_f)T - mN_f}{2r-N_f}\right)^{N_f-r}.\end{aligned} \qquad (4.15)$$

According to eq. (4.14), $R \sim T^2 + \cdots$ and thus the order eq. (4.15) in T is $2N_c - N_f$. Therefore, there are $2N_c - N_f$ solutions.

To get the total number of solutions, we must take in account two things. There is a degeneracy in choosing the sign in front of the square root for each flavor. This accounts for a factor of 2^N_f. Another one is the symmetry of eq. (4.15) under $r \to N_f - r$. The effect of the symmetry will be dependent on whether N_f is even or odd. In the even case, the solutions will come in pairs. In the odd case, a half number of solutions of T will not be consistent with the sign choice which was lost when (4.14) was squared to determine R. Therefore, in any case, it cuts the number of solutions by a half. We conclude now that the total number of solutions is $(2N_c - N_f) \cdot 2^{N_f - 1}$ [27, 21].

In the case when N_f is even and $r = N_f/2$, we obtain $T = 0$ and eq. (4.15) reduces to $R^{N_c - N_f/2} = \Lambda^{2N_c - nf}$. Again, we get $(2N_c - N_f)/2$ solutions and the counting is the same as above.

In the previous section, we showed that the r-th non-baryonic branches of the (N_c, N_f) theory correspond to the baryonic branch of $(\tilde{N}_c, \tilde{N}_f)$ theory. In particular, the Seiberg-Witten curve factorizes as eq. (4.12). Now, $P_{\tilde{N}_c}(x, \tilde{u}_k)^2 - 4\Lambda^{\tilde{N}_c} \prod_{k=r+1}^{N_f}$ must be factorized as the $r = 0$ case of $(\tilde{N}_c, \tilde{N}_f)$ theory. This has $2\tilde{N}_c - \tilde{N}_f = 2N_c - N_f$ solutions.

Now, once again, we note the symmetry under $r \to N_f - r$. This restricts $r \leq [N_f/2]$. Also, there is a degeneracy for each r, in choosing r $(x+m)$ factors out of N_f masses which are each equal to m. In the case of odd N_f, the counting is now complete as

$$(2N_c - N_f) \sum_{r=0}^{(N_f-1)/2} \binom{N_f}{r} = (2N_c - N_f) \cdot 2^{N_f - 1}\,.$$

When N_f is even, the case $r = N_f/2$ is subtle. It has only $(N_c - N_f/2)$ solutions. However, total number of solutions is once again

$$(2N_c - N_f) \sum_{r=0}^{(N_f-1)/2} + (N_c - N_f/2) \binom{N_f}{N_f/2} = (2N_c - N_f) \cdot 2^{N_f - 1}\,.$$

In fact, as readers may have noticed, our counting is in essence the same as in [27].

Our counting is consistent with the examples in section 3.1 and 3.2. In eq. (3.15), we discovered $(2 \cdot 2 - 2) \cdot 2 = 4$ solutions. It would seem that eq. (3.23) has only three solutions, but $u_2 = m^2 + \Lambda^2$ is a doubly degenerate solution as we noted. For SU(3), we expect $(2 \cdot 3 - 3) \cdot 2^2 = 12$ solutions, as we found in section 3.2.

5. M-theory construction

As discussed in the Introduction, a non-baryonic Higgs branch opens up when a pair of flavor masses become equal. Furthermore, as discussed in section 4, the effective superpotential in the non-baryonic vacua of the $\mathcal{N}$=1 theory matches the baryonic vacua of the theory with one fewer colors and two fewer flavors. We can easily understand this from the M5-brane construction of these theories. Witten has demonstrated that the Seiberg-Witten curve of an $\mathcal{N} = 2$ gauge theory can often be determined as the geometry of the M5-brane configuration whose low energy dynamics is described by the gauge theory [30]. The Type IIA brane configuration for the $\mathcal{N} = 2$ SU(N) gauge theory with N flavors corresponds to

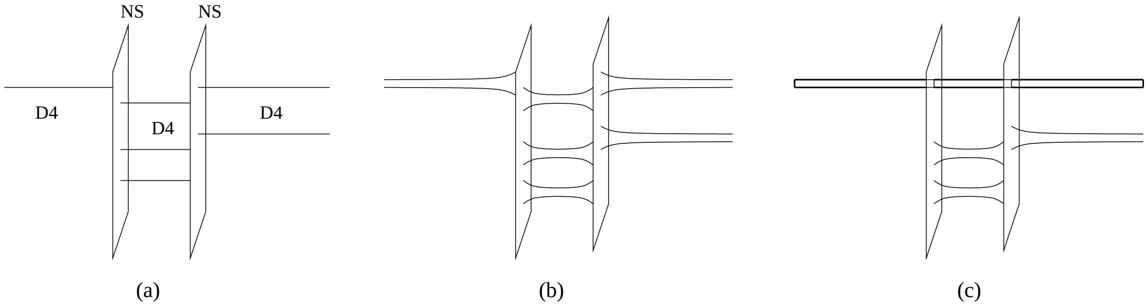

Figure 2: (a) Type IIA brane construction of $\mathcal{N}=2$ theory. (b) M5-brane configuration of same theory. (c) Seiberg-Witten curve factorizes at non-baryonic root of the Higgs branch.

a stack of N D4-branes stretched between a pair of NS5-branes, with N semi-infinite D4-branes attached to each of the NS5-branes [31], as in figure 2a. The positions of the finite D4-branes correspond to the adjoint vevs, and the positions of the semi-infinite D4-branes correspond to the hypermultiplet masses. In the M-theory picture, the entire setup is described by a single M5-brane, in which the D4-branes correspond to the M5-brane wrapped around the M-theory circle, and the NS5-branes correspond to the M5-brane at a point on the M-theory circle, as in figure 2b. A factor of $(x-e_i)^2$ appears in the Seiberg-Witten curve for each hypermultiplet that becomes massless We can set pairs of hypermultiplet masses equal by matching the positions of the intersection of the semi-infinite D4-branes with the NS5-branes on opposite NS5-branes. In this way, when one of the D4-branes suspended between the NS5-branes aligns itself with that pair of semi-infinite D4-branes, they merge and become one large D4-brane (figure 2c) which can then be removed from the rest of the brane configuration in a transverse direction without any cost of energy. This represents a non-baryonic root of the moduli space. When the Coulomb and Higgs branches meet, the M5-brane "tube" can be removed, and the algebraic curve describing the M5-brane factorizes. An $\mathcal{N}=2$ nonrenormalization theorem implies that the Higgs branch and the Coulomb branch are decoupled [18], so after factoring out the tube, what remains of the M5-brane geometry must be the Seiberg-Witten curve of the Higgsed theory with one fewer colors and two fewer flavors, as we found evidence for in section 4. The baryonic branch corresponds to the joining of each of the finite D4-branes with a semi-infinite D4-brane, which because of the freezing of the U(1) can only happen if the average position of both types of D4-branes is the same, in which case the NS5-brane at the junction can be removed from the configuration.

6. Conclusions

We have extended previous predictions for the vacuum structure of broken $\mathcal{N}=2$ gauge theories with matter to the case of SU(N_c) gauge group with equal number of flavors and colors. Below the scale of the massive adjoint hypermultiplet this theory acquires a quantum modified constraint on the moduli due to nonperturbative effects. With this constraint we determined the field theory expectation for the low energy superpotential below the lowest mass scale of the theory. We found a number of solutions corresponding to the various vacua of the theory. These results were compared with the matrix model prediction for the low energy superpotential and were found to agree. Although the presence of baryons in the theory could potentially have been a problem for the naive generalization of the matrix model results for $N_f < N_c$ to this case, we argued that the low energy superpotential is independent of the baryon vevs. We studied the SU(2) and SU(3) theories in detail and verified the matrix model prediction for the location on the Coulomb branch at which the Seiberg-Witten curve maximally factorizes, with $(N_c - 1)$ pairs of double roots. It would be interesting to extend these results to theories in which the baryons play a nontrivial role, for example by adding a tree level superpotential depending on them.

We also demonstrated a relation between the nonbaryonic vacua of the SU(N_c) theory with N_f flavors and specially tuned masses and the vacua of the SU$(N_c - r)$ theory with $N_f - 2r$ flavors. To be precise, we checked that one of the moduli (u_2) matches between these vacua, and by the $\mathcal{N} = 2$ nonrenormalization theorem we expect the same to be true for the other moduli. It would be nice to check this claim explicitly via the matrix model solutions.

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